Higher Order Stochastic Dominance and Moral Hazard

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Abstract

In this paper we study higher order risk effects in a classic model of moral hazard. We argue that an increase in the level of effort exerted by the agent can produce not only first-order stochastic dominant shifts in the distribution of results, but also higher order stochastic changes that are relevant for some contractual situations that we will describe in this paper. We discuss the economic intuition behind these situations and we also state the mathematical conditions required in equilibrium for second and third-order stochastic dominant shifts to provide a right ordering of risky prospects.

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1 Introduction

Moral hazard models have become a prominent tool to analyze a wide range of economic situations. Moral hazard models were first used in insurance markets (Borch [1962], Zeckhauser [1970]), in economic analyses of incentives and authority within organizations (Mirrlees, 1976), and also in the study of financial markets (Jensen and Meckling [1976], Myers [1977] and Ross [1977]). But, as the academic community observed the usefulness of the approach, other subfields of economics soon began to incorporate moral hazard setups into their analyses. Today, from accounting to labor relations, from macroeconomics to corporate finance, most economic specialties have included moral hazard as part of their theoretical framework of analysis.

One key idea found in moral hazard problems is that the principal always wants the agent to exert more effort than the real effort exerted in the second-best equilibrium. The reason is very simple: "the greater the effort exerted by the agent, the better the distribution of results turns out". Therefore, defining the changes in the conditional distribution of results when the agent's effort changes is a key element in the modeling of moral hazard problems. The first way to think about the effects of agent's effort on the distribution of results is to consider first order stochastic dominant (FOSD) shifts. In FOSD, as long as the utility function of an economic agent is increasing and continuous, if distribution A is always preferred over distribution B, then distribution A FOSD distribution B. FOSD is a universal assumption used in economic problems that rest in moral hazard as their main analytical tool.

But FOSD shifts are not the only way to represent **a better distributions of outcomes**. When economic agents are highly risk averse, second-order stochastic dominance (SOSD) may be more important in some cases. For example, let us suppose that the principal is delegating the management of a portfolio to an agent. If the principal is about to retire and she wants her savings to be managed in a way that minimizes risk (or volatility) instead of the usual assumption of maximizing expected returns, the principal wants the increase in the agent's effort to translate into safer investments (decreasing risk), which certainly can be described by SOSD shifts instead of FOSD shifts. By way of illustration, a special case of SOSD shift is a mean-preserving decrease in risk described by Rothschild and Stiglitz (1970).

But we have yet more alternatives. Consider the case where the principal's primary objective is to avoid bad states of nature. If this were the case, the principal can be characterized as an individual who exhibits downside risk aversion (Menezes et. al, 1980). Consequently, the principal will be extremely **loss averse** (Kahneman and Tversky [1979]). Downside risk aversion implies that the principal is **prudent** (see Kimball [1990]) and therefore, third-order stochastic dominant (TOSD) shifts will provide the right ordering of risky prospects.

Another example is when the principal is temperate and thus exhibits aversion to outer risk (see for instance Eeckhoudt, Gollier and Schneider [1995], Menezes and Wang [2005]). In this case, fourth-order stochastic dominant (FOOSD) shifts will provide the right ordering of distributions. We can continue for higher order risk changes infinitely. For interesting applications of higher order risk models see for example Jindapon and Nielson (2007) or Eeckhoudt and Schlesinger (2013).

In this paper we want to discuss how higher order stochastic changes can be incorporated into traditional moral hazard problems. In the last twenty years we have seen important developments in the economic theory of risk and we believe that incorporating these new developments into traditional applied economic models that have basic risk structures is one way of enriching those models. We show that higher order stochastic dominance is a key **necessary** condition to rank prospects in higher order risk attitude settings. However, an additional condition is required in order to obtain definitive results, and that condition relates to the sign of higher order derivatives of the principal's optimal share, as we will see below.

2 The Model

Following Hölmstrom (1979), we study a principal-agent problem where the agent privately takes an action (or effort level) a from set $A \sqsubseteq \mathbb{R}$. The monetary outcome or payoff $\widetilde{x}$ is random and let $x \in X \sqsubseteq \mathbb{R}$ denote an outcome. Without loss of generality, we assume that X is an interval of the form $[\underline{x}, \overline{x}]$, with $\underline{x} < \overline{x}$ and $(\underline{x}, \overline{x}) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$. The agent first decides of her action a and then both the principal and the agent observe outcome x. We denote by $F(\cdot \mid a)$ the cumulative distribution function of $\widetilde{x}$, given action a taken by the agent. In addition, we assume that the support of F is independent of a and there is exists a conditional density function denoted by $f(\cdot \mid a)$ such that $f_a(x \mid a)$ and $f_{aa}(x \mid a)$ are well-defined for all (x, a). Finally, let $s(x)$ be the share of the payoff received by the agent given payoff x and $r(x) = x - s(x)$ the share of the payoff received by the principal. Following Hölmstrom (1979), given outcome x, we restrict the sharing rule $s(x)$ to lie in some interval I of the form $[c, x + d]$ in order to avoid non-existence of a solution.

2.1 The Agent

The agent's preferences are represented by a separable expected utility function that is given by $\int u(s(x)) f(x \mid a)\,dx - c(a)$, where u is a Von Neumann-Morgenstern utility function with $u' > 0$, $u'' < 0$ and c is a convex cost function that represents the disutility of the agent's effort; we assume that $c' \geq 0$ and $c'' \geq 0$. In addition, the agent has an outside job opportunity that provides reservation utility u_0.

Given a sharing rule s, the agent aims to maximize her expected utility

$$\max_{a \in A} \int_X u(s(x)) f(x \mid a)\,dx - c(a).$$

Assuming an interior solution, the optimal effort a^* she exerts must be such that

$$\int_X u(s(x)) f_a(x \mid a^*)\,dx - c'(a^*) = 0,$$

which represents her optimal choice of a given s.

2.2 The Principal

Given an action a taken by the agent, the principal's expected utility function is given by

$$\int_X G(x - s(x)) f(x \mid a)\,dx,$$

where G is a also a Von Neumann-Morgenstern utility function with $G' \geq 0$, $G'' \leq 0$, $G''' \geq 0$ and $G^{iv} \leq 0$. That is, the principal exhibits positive marginal utility, risk aversion, a prudent behavior and aversion to outer risk (or temperance behavior).

The principal's optimal decision problem is:

$$\begin{array}{lll} \max\limits_{\{s \in I,\, a \in A\}} & \int_X G(x - s(x)) f(x \mid a)\,dx & \\ \text{s.t.} & \int_X u(s(x)) f(x \mid a)\,dx - c(a) \geq u_0 & \text{(C1)} \\ & \int_X u(s(x)) f_a(x \mid a)\,dx - c'(a) = 0 & \text{(C2)} \end{array}$$

Let λ and μ denote the Lagrange multipliers of constraints (C1) and (C2) respectively. Constraint (C1) represents the participation constraint (or the individual rationality constraint) whereas constraint (C2) is the agent's incentive compatibility constraint, assuming an interior solution.

2.3 Optimality Conditions

Using pointwise optimization, the first order condition of the Lagrangian function L with respect to s is given by:

$$\frac{\partial L}{\partial s} = -G'(x - s(x)) f(x \mid a) + \lambda u'(s(x)) f(x \mid a) + \mu (u'(s(x)) f_a(x \mid a)) = 0,$$

which turns into

$$\frac{G'(x - s(x))}{u'(s(x))} = \lambda + \mu \frac{f_a(x \mid a)}{f(x \mid a)}. \tag{1}$$

We will refer to relationship (1) as the equation that provides the optimal sharing rule under *moral hazard*, or equivalently, as the *second best* case. When there is no asymmetric information, the principal observes a and can contract directly based on a. In this case, we do not need the incentive compatibility constraint and therefore, the optimality condition of equation (1) becomes

$$\frac{G'(x - s(x))}{u'(s(x))} = \lambda^*. \tag{2}$$

This optimal sharing rule is called the *Borch rule* and we will refer to it as the *Pareto optimal solution* or as the *first best* case. Obviously, λ^* from equation (2) does not need to be the same as λ from equation (1). Going back to our original problem, the first order condition of the Lagrangian with respect to a is given by

$$\begin{aligned}\frac{\partial L}{\partial a} = & \int_X G(x - s(x)) f_a(x \mid a)\, dx + \lambda \left(\int_X u(s(x)) f_a(x \mid a)\, dx - c'(a)\right) \\ & + \mu \left(\int_X u(s(x)) f_{aa}(x \mid a)\, dx - c''(a)\right) = 0\end{aligned}$$

which turns into

$$\mu = -\frac{\int_X G(x - s(x)) f_a(x \mid a)\, dx}{\int_X u(s(x)) f_{aa}(x \mid a)\, dx - c''(a)}. \tag{3}$$

Notice that $\int_X u(s(x)) f_{aa}(x \mid a)\, dx - c''(a)$ is the second order condition of the agent's optimal effort and therefore, it is always negative. If $\mu \geq 0$, it means that the principal wants the agent to exert more effort than the actual effort exerted in the second best equilibrium. But as we have argued, more effort can translate into higher order stochastic dominant shifts in the distribution of results. For now, observe that in order to have $\mu \geq 0$, we only need $\int_X G(x - s(x)) f_a(x \mid a)\, dx$ **to be positive**. We will come to this point later in the paper. Now, let us introduce the concept of stochastic dominance of different order because these concepts are needed to interpret the main results of this paper.

2.4 Stochastic Dominance

2.4.1 Preliminary Results

We shall use the definition given by Ekern (1980) to introduce the concept of N^{th} order stochastic dominance. For $x \geq \underline{x}$, let us define $F^{k}(x \mid a) = \int_{\underline{x}}^{x} F^{k-1}(y \mid a)\, dy$, for $k = 1, 2, ..., N$. Note that $F^{1}(x \mid a) = F(x \mid a) = \int_{\underline{x}}^{x} F^{0}(y \mid a)\, dy$ and by definition $F^{0}(\cdot \mid a)$ is the density function $f(\cdot \mid a)$. Also, $F(\underline{x} \mid a) = 0$ and $F(\overline{x} \mid a) = 1$ for all $a \geq 0$. By iteration, we set

$$
\begin{aligned}
F^{2}(x \mid a) &= \int_{\underline{x}}^{x} F(y \mid a)\, dy \\
F^{3}(x \mid a) &= \int_{\underline{x}}^{x} F^{2}(y \mid a)\, dy,
\end{aligned}
$$

and so on, until obtaining the general formulation with N integrals of the form

$$F^{N}(x \mid a) = \int_{\underline{x}}^{x} F^{N-1}(y \mid a)\, dy.$$

2.5 Mixed Risk Aversion and Risk Aversion of N^{th} Order

In this section, we assume that the set of outcomes X is a subset of $\mathbb{R}_{+}$.

2.5.1 Mixed Risk Aversion

An individual is said to exhibit "mixed risk aversion" if her preferences are represented by a $C^{\infty}(\mathbb{R}_{+})$ utility function u such

$$(-1)^{n} u^{(n)}(z) < 0,$$

for all $n \in \mathbb{N}$ and $z \in \mathbb{R}_{+}$. Then, if $(-1)^{n} u^{(n+1)} > 0$, for all $n \in \mathbb{N}$, we define $A_{n} = -\frac{u^{(n+1)}}{u^{(n)}}$ as the coefficient of absolute risk aversion of order n. Observe that for "mixed risk averse" individuals, we have $A_{n} > 0$, for all $n \in \mathbb{N}$. A_{1} is the Arrow-Pratt coefficient of risk aversion, $A_{2} > 0$ is the prudence ratio and measures precautionary saving motives (see Kimball [1990]). Finally, A_{3} is the temperance ratio. Following Eeckhoudt, Gollier and Schneider (1995), an individual is said to be temperate, i.e. marginal gains in expected utility for successive upwards shifts of any increase in risk[1] are decreasing. For temperate individuals, we have $A_{3} > 0$.

[1]Broadly speaking, an upwards shift of an increase in risk (SIR) induces more risk in good (wealthier) states and less risk elsewhere. A SIR preserves both the mean and the variance of a distribution.

It is easy to verify that if utility function u exhibits "mixed risk aversion", then the marginal utility function u' is completely monotone[2], then utility function u exhibits "mixed risk aversion". Then, by Hausdorff-Bernstein-Widder's Theorem (see Bernstein [1928], Hausdorff [1 and Widder [1931]), a representation of u' is given by

$$u'(x) = \int_0^\infty e^{-xs} dF(s),$$

where F is a distribution function on $\mathbb{R}_+$. This means that u' is the Laplace transform of F. It is well known that for all $n \in \mathbb{N}$ the limit $\lim\limits_{0^+} u^{(n)}$ exists but is not necessary finite; more specifically, $\lim\limits_{0^+} u^{(n)}$ is finite if and only if $\int_0^\infty s^{n-1} dF(s) < \infty$. In addition, as shown in Caballé and Pomansky (1996), a representation of the utility function u is given by

$$u(x) = u(0) + \int_0^\infty \frac{1 - e^{-xs}}{s} dF(s),$$

where distribution F is such that $\int_1^\infty \frac{dF(s)}{s} < \infty$. Finally, the following proposition provides a characterization of "mixed risk aversion" in terms of coefficient of absolute risk aversion of order n.

Proposition 1

Assume that utility function u is $C(\mathbb{R}_+) \cap C^\infty(\mathbb{R}_{++})$, with $u' > 0$ such that for all $x \in \mathbb{R}_+$, $u^{(n+1)}(x) \neq 0$. Then u exhibits "mixed risk aversion" if and only if for all $n \in \mathbb{N}$, A_n is a positive non-decreasing sequence.

Proof: Step 1**:** $\Rightarrow$ Assume that $(-1)^n u^{(n+1)} > 0$ so that for all $x \in \mathbb{R}_{++}$ we can write

$$u'(x) = \int_0^\infty e^{-xs} dF(s),$$

for some distribution function F that is not identically equal to zero. It follows that

$$u^{(n)}(x) = (-1)^{n-1} \int_0^\infty s^{n-1} e^{-xs} dF(s),$$

and therefore

$$A_n(x) = \frac{\int_0^\infty s^n e^{-xs} dF(s)}{\int_0^\infty s^{n-1} e^{-xs} dF(s)}.$$

[2] A $C(\mathbb{R}_+) \cap C^\infty(\mathbb{R}_{++})$ function f is said to be completely monotone if for all $k \in \mathbb{N}$ and $z \in \mathbb{R}_+$, $(-1)^k f^{(k)}(z) \geq 0$.

To show that $A_{n+1} \geq A_n$ it is equivalent to prove that

$$\int_0^\infty s^{n+1}e^{-xs}dF(s) \times \int_0^\infty s^{n-1}e^{-xs}dF(s) \geq \left(\int_0^\infty s^n e^{-xs}dF(s)\right)^2.$$

Consider the quadratic Q in λ

$$Q(\lambda) = \int_0^\infty (\lambda s+1)^2 s^{n-1}e^{-xs}dF(s).$$

Clearly, Q is a non-positive function so at most it has a double real root; therefore, its discriminant Δ must non-positive, i.e.,

$$4\left(\int_0^\infty s^n e^{-xs}dF(s)\right)^2 - 4\int_0^\infty s^{n+1}e^{-xs}dF(s) \times \int_0^\infty s^{n-1}e^{-xs}dF(s) \leq 0.$$

This concludes step 1.

Step 2: $\Leftarrow$ Conversely, assume that A_n is strictly increasing and positive. For $n = 1$, we have

$$A_1 = -\frac{u''}{u'} > 0,$$

so $u'' < 0$ as $u' > 0$ (by assumption) and for $n = 2$, we have

$$-\frac{u'''}{u''} > -\frac{u''}{u'} > 0,$$

so we must have $u''' > 0$. We now show the result by induction on n. Assume that $(-1)^n u^{(n+1)} > 0$. We write

$$A_{n+1} = -\frac{u^{(n+2)}}{u^{(n+1)}} = \frac{(-1)^{n+1}u^{(n+2)}}{(-1)^n u^{(n+1)}} > A_n > 0.$$

As $(-1)^n u^{(n+1)} > 0$ we deduce that $(-1)^{n+1}u^{(n+2)} > 0$, which concludes step and the proof is complete.

Proposition 1 implies that A_n admits a limit (not necessarily) finite as n goes to ∞. Furthermore, for negative exponential utility functions, coefficient A_n is constant for all $n \in \mathbb{N}$. so even if $(-1)^n u^{(n+1)} > 0$ for all $n \in \mathbb{N}$, A_n may not be an increasing sequence. Finally, observe that A_n is a decreasing function of x since

$$\begin{aligned} A_n'(x) &= -\frac{u^{(n+2)}(x)}{u^{(n)}(x)} + \left(\frac{u^{(n+1)}(x)}{u^{(n)}(x)}\right)^2 \\ &= -A_n(x)\left[A_{n+1}(x) - A_n(x)\right] < 0. \end{aligned}$$

This last result was first derived by Caballé and Pomansky (1996).

2.5.2 Risk Aversion of N^{th} Order

Let u be a $C(\mathbb{R}_+) \cap C^N(\mathbb{R}_{++})$ utility function and $u^{(k)}$ its k^{th} derivative, for $k = 1, 2, ..., N$. Following Ekern (1980), an individual is said to exhibit N^{th} order risk aversion if

$$(-1)^N u^{(N)}(x) < 0, \text{ for } 0 \leq x \leq \infty. \tag{4}$$

Function u' is $N+1$-times monotone[3] on $\mathbb{R}_{++}$ and by Williamson's theorem (1956), u' admits the following representation

$$u'(x) = \int_0^\infty \left([1 - xs]^+\right)^N dF(s),$$

for some distribution function F and where $[x]^+ = \max\{x, 0\}$ denotes the positive part of real number x.

2.5.3 Stochastic Dominance of N^{th} Order

Let F_1 and F_2 be two cumulative probability functions. We say the distribution F_1 N^{th} order stochastic dominates F_2 if

$$\begin{aligned} F_1^k(\overline{x}) &\leq F_2^k(\overline{x}), \text{ for } k = 1, 2, ..., N-1 \\ F_1^N(x) &\leq F_2^N(x), \text{ for } x \in X, \end{aligned}$$

with at least one strict inequality.

2.5.4 Application to Moral Hazard Problem

Using these definitions, we observe that in our moral hazard model, the distribution of results shifts when agent's effort level a changes, and given that a is a continuous variable, we can use an extended version of the Diamond and Stiglitz (1974)'s definition of a mean-preserving increase (decrease) in risk but for higher order risk changes in order to accommodate our problem to a differentiable setting on agent's effort. For our particular purposes, the key element to have in

[3] A function $f \in C(\mathbb{R}_+) \cap C^{k-2}(\mathbb{R}_{++})$, $k \geq 2$, for which $(-1)^k f^{(l)} \geq 0$, non-increasing and, convex for $l = 0, 1, 2, ..., k-2$ is called k-times monotone on $\mathbb{R}_{++}$. In the case $k = 1$, we only require f to be non-negative and non-increasing.

mind is the partial derivative of the repeated integral $F^k(x \mid a)$, which is denoted by $F_a^k(x \mid a)$. As a result, FOSD will be associated with $F(x \mid a_2) - F(x \mid a_1) \leq 0$, with $a_1 \leq a_2$, which implies that $F_a(x \mid a) \leq 0$ in the moral hazard model, SOSD will be associated with $F_a^2(x \mid a) \leq 0$, TOSD will be associated with $F_a^3(x \mid a) \leq 0$ and so on. The following definition formalizes these concepts.

An increase in agent's effort a represents an Nth degree stochastic dominant (NSD) shift if $F_a^N(x \mid a) \leq 0$ for all $x \in X$, where inequality is strict for some x, and $F_a^k(\overline{x} \mid a) \leq 0$ for $k = 1, 2, ..., N-1$.

An increase in agent's effort a represents an Nth degree decrease in risk if $F_a^N(x \mid a) \leq 0$ for all $x \in X$ where inequality is strict for some x, and $F_a^k(\overline{x} \mid a) = 0$ for $k = 1, 2, ..., N-1$.

The intuition behind definitions 1 and 2 is straightforward when we use a discrete version of the effort model. Following Ekern (1980), let $\widetilde{x}_1$ and $\widetilde{x}_2$ be two random variables with a probability distribution given by $F(\cdot \mid a_1)$ and $F(\cdot \mid a_2)$ respectively, with $a_1 < a_2$. The condition $F_a^k(\overline{x} \mid a) = 0$ is equivalent to $F^k(\overline{x} \mid a_1) = F^k(\overline{x} \mid a_2)$ for all $k = 1, 2, ..., N-1$ in the discrete model. This means that the $N-1$ moments of both distributions are the same. For example, we can say that $\widetilde{x}_2$ is a second-degree decrease in risk of $\widetilde{x}_1$ if $\widetilde{x}_2$ dominates $\widetilde{x}_1$ via second order stochastic dominance and both distributions have equal mean, which is the case of the Rothschild and Stiglitz (1970) mean-preserving decrease in risk. A third-degree decrease in risk on the other hand, is equivalent to the concept of a decrease in downside risk developed by Menezes et al. (1980). Similarly, a fourth-degree decrease in risk is equivalent to the concept of a decrease in outer risk as developed by Menezes and Wang (2005), and so on.

2.5.5 N^{th} Degree Stochastic Change Equivalence

The last idea from the economics of risks literature that we will use in our model is the concept of N^{th} degree stochastic change equivalence. We will make use of this concept in the final discussion of the present paper. Let V be a utility function that is differentiable at least N times on the interval X. The expectation of V is defined as $E(V(\widetilde{x})) = \int_{\underline{x}}^{\overline{x}} V(x)dF(x \mid a)$. After N iterations of integrating by parts, we find that

$$E(V(\widetilde{x})) = \sum_{k=1}^{N} (-1)^k V^{(k-1)}(\overline{x}) F^k(\overline{x} \mid a) + \int_{\underline{x}}^{\overline{x}} (-1)^N V^{(N)}(x) F^N(x \mid a) dx. \tag{5}$$

To evaluate the effect of a change in agent's effort a on the expected utility, we differentiate relationship (5) with respect to a to obtain

$$\int_{\underline{x}}^{\overline{x}} V(x)dF_a(x \mid a) = \sum_{k=1}^{N}(-1)^k V^{(k-1)}(\overline{x})F_a^k(\overline{x} \mid a) + \int_{\underline{x}}^{\overline{x}}(-1)^N V^{(N)}(x)F_a^N(x \mid a)dx.$$

An increase in the agent's effort a that represents a NSD shift of the distribution F is equivalent to $\int_{\underline{x}}^{\overline{x}} V(x)dF_a(x \mid a) \geq 0$, if and only if $(-1)^k V^{(k)}(x) \leq 0$ for $k = 1, 2, ..., N$ and $x \in X$.

3 Stochastic Dominance and Moral Hazard

3.1 FOSD and Moral Hazard

The usual way that moral hazard problems with continuous effort are solved is by using the first order condition given by Hölmstrom (1979). However, to be able to use Hölmstrom's conditions, some mathematical properties of the functions involved in the problem must be satisfied, such the convexity of the distribution function and the monotone likelihood ratio condition (Rogerson (1985)), and given that we are using only general functional forms in this paper, we will assume that all these technicalities are met and therefore, we can focus on the stochastic shifts of different order of the distribution of results. For the FOSD case we could have referred the reader to the original source without any exposition here, but instead, we believe that it is important to provide an alternative and somewhat simplified version of Hölmstrom's proof in order to highlight the importance of FOSD in the agency literature. Then, we will build on this to analyze higher order stochastic dominant shifts.

Proposition 2 ***(Hölmstrom, [1979])***

If an increase in agent's effort corresponds to a FOSD shift of the distribution F, i.e. a shift that satisfies the property of $F_a^2(\cdot \mid a) \leq 0$, the principal would always desire a greater effort from the agent than the latter would be willing to provide under the second best sharing rule, i.e. $\mu > 0$.

Proof: By contradiction. Let us suppose the opposite, i.e. $\mu < 0$. In this case, the principal's marginal expected utility is higher in the moral hazard case than in the Pareto case. Let us denote s^* the agent's payment in the Pareto case and observe that by (2), $s^*(x)$ and $x - s^*(x)$ are increasing in x. Then, when $\mu < 0$ relationship (1) guarantees that $s(x) \leq s^*(x)$

on the set $\{x,\ f_a(x \mid a) > 0\}$, and $s(x) > s^*(x)$ on the set $\{x,\ f_a(x \mid a) < 0\}$. Consequently, we have

$$\int_X G(x - s(x)) f_a(x \mid a)\, dx \geq \int_X G(x - s^*(x)) f_a(x \mid a)\, dx. \tag{6}$$

Integrating by parts the RHS of inequality (6) implies

$$\textstyle\int_X G(x - s^*(x)) f_a(x \mid a)\, dx = \quad G(x - s^*(x)) F_a(x \mid a) \Big|_{\underline{x}}^{\overline{x}} - \int_X G_x(x - s^*(x)) F_a(x \mid a)\, dx,$$

and since $F_a(\underline{x} \mid a) = F_a(\overline{x} \mid a) = 0$ for any a, we have that

$$\int_X G(x - s^*(x)) f_a(x \mid a)\, dx = -\int_X G_x(x - s^*(x)) F_a(x \mid a)\, dx, \tag{7}$$

which is negative for $\mu < 0$ by relationship (3). This is equivalent to say that

$$\int_X G_x(x - s^*(x)) F_a(x \mid a)\, dx > 0, \text{ for } \mu < 0.$$

Set $r^*(x) = x - s^*(x)$. We know that $G_x(r^*(x)) = G'(r^*(x))(r^*)'(x)$ and $G' > 0$. To obtain the sign of $r^{*\prime}(x)$ we differentiate with respect to x relationship (1) to obtain

$$\frac{G''(r^*(x))\, r^{*\prime}(x)\, u'(x - r^*(x)) - G'(r^*(x))\, u''(x - r^*(x))\,[1 - r^{*\prime}(x)]}{[u'(x - r^*(x))]^2} = 0,$$

which can be re-arranged as

$$r^{*\prime}(x) = \frac{G'(r^*(x))\, u''(x - r^*(x))}{G''(r^*(x))\, u'(x - r^*(x)) + G'(r^*(x))\, u''(x - r^*(x))},$$

so that

$$r^{*\prime}(x) = \frac{A(r^*(x))}{A(r^*(x)) + P(s^*(x))} > 0. \tag{8}$$

where A and P denote the Arrow-Pratt absolute risk aversion coefficient of the agent and principal respectively.

Therefore, having established that $r^{*\prime}(x) > 0$, since $G'(r^*(x)) > 0$ and by FOSD $F_a(x \mid a) < 0$, we conclude that

$$\int_X G_x(x - s^*(x)) F_a(x \mid a)\, dx < 0,$$

which is a **contradiction**. Therefore, μ must be positive, and consequently, when the agent's effort induces FOSD shifts in the distribution of results, the principal will always want the agent to exert more effort than the latter would be willing to provide under a moral hazard setting. The proof is complete. ■

The previous proof relies on the concept of FOSD. But, what if the agent's effort affects the distribution of results through higher order stochastic dominance changes? FOSD is only one alternative of a stochastic change; it may even be the simplest way to think in the effect of agent's effort on the distribution of results. However, higher order stochastic changes can also make economic sense in other contexts as we will see below.

3.2 SOSD and Moral Hazard

As we discussed above, FOSD is a reasonable way to represent the effects of agent's effort on the distribution of results. However, in some cases, SOSD could be a better way to represent the effect of an agent's effort on a delegation set up under moral hazard. For instance, when a **risk averse** principal needs to delegate the management of a portfolio to an agent, it may be the case that the principal is very sensitive to the dispersion of a distribution and in consequence, SOSD shifts may provide the right preference ordering of distributions. For example, when a principal is about to retire (short-run horizon of investment) and he is not willing to take risky bets, the opposing interests between the principal and the agent come not only from the effort-payment combination they have to agree upon, but also from the potentially different valuation of the risk-return trade off they may have. In this case, a very risk-averse principal may want to minimize risk, and the optimal contract has to be designed in a way that consider agent's effort cost and also her valuation of the risk-return trade off.

To our knowledge, Hughes (1982) is the only application of agency theory under SOSD shifts induced by increasing agent's efforts. In this subsection, we generalize the results of Hughes (1982) that are based on Harmonic Absolute Risk Aversion (HARA) preferences, which are preferences that exhibit linear absolute risk tolerance. We use instead a general formulation of absolute risk tolerance to show that SOSD is the key property that defines the optimal contract when the principal is mainly interested in avoiding disperse distributions of outcomes.

Proposition 3 ***(Generalizing Huges [1982])***

If an increase in the agent's effort corresponds to a SOSD shift of the distribution F, i.e. a shift that satisfies the property of $\int^{\cdot} F_a\left(y \mid a\right) dy \leq 0$, the principal would always want greater effort of the agent than the agent would be willing to provide under the second-best sharing rule, i.e. $\mu > 0$.

Proof: Everything remains the same as in Proposition 2, except that this time we have to incorporate SOSD shifts into the analysis. To do this we have to integrate by parts a second time the right hand side of relationship (7) and show that the resulting expression is **positive**. Then, integrating by parts the expression $-\int_X G_x(x - s^*(x)) F_a(x \mid a)\, dx$ while keeping in mind that $r^*(x) = x - s^*(x)$, we obtain

$$\begin{gathered} -\int_X G_x(r^*(x)) F_a(x \mid a)\, dx = \\ -G_x(r^*(x)) \int^x F_a(y \mid a)\, dy \Big|_{\underline{x}}^{\overline{x}} + \int_X \left[G_{xx}(r^*(x)) \int^x F_a(y \mid a)\, dy\right] dx. \end{gathered} \tag{9}$$

Then, relationship (9) transforms into

$$\begin{gathered} \int_X G(x - s^*(x)) f_a(x \mid a)\, dx = \\ -G_x(r^*(\overline{x})) F_a^2(\overline{x} \mid a) + \int_X \left[G_{xx}(r^*(x)) \int^x F_a(y \mid a)\, dy\right] dx. \end{gathered} \tag{10}$$

The definition of SOSD ensures that $F_a^2(\overline{x} \mid a) \leq 0$ and $\int F_a(x \mid a)\, dx \leq 0$, and given that $G_x(r^*(\overline{x}))$ is positive, to sign relationship (10), we focus on the sign of G_{xx}. From $G_x(r^*(x)) = G'(r^*(x))\, r'(x)$, we can obtain the expression for G_{xx}, and that is

$$G_{xx}(r^*(x)) = G''(r^*(x)) \cdot (r^{*\prime}(x))^2 + G'(r^*(x)) \cdot r^{*\prime\prime}(x).$$

Since $(r^{*\prime}(x))^2 > 0$, $G'' < 0$, we know that $G''(r^*(x)) \cdot (r^{*\prime}(x))^2 < 0$. Also, since $G'(r^*(x)) > 0$ we only need that $r^{*\prime\prime}(x)$ to be negative to obtain a negative $G_{xx}(r^*(x))$ in order to have a positive $\int_X \left[G_{xx}(r^*(x)) \int^x F_a(y \mid a)\, dy\right] dx$ and this way to be sure that the RHS of relationship (7) is positive. Next, let us examine the sign of $r^{*\prime\prime}(x)$. Let us recall that by relationship (8), we know that $0 < r^{*\prime}(x) = \dfrac{A}{A+P} < 1$.

Gollier (2001) defines the absolute risk tolerance index as the inverse of the absolute risk aversion coefficient, that is $T = \dfrac{1}{A}$. When the preferences are DARA, an increase in wealth implies a decrease in the absolute risk aversion coefficient, which in tuns implies an increase in the absolute risk tolerance index. That is $T_A' > 0$ and $T_P' > 0$. With this definition in mind, we can rewrite relationship (8)

$$r^{*\prime}(x) = \frac{T_P(r^*(x))}{T_A(s^*(x)) + T_P(r^*(x))}, \tag{11}$$

So, in order to obtain $r^{*\prime\prime}$, we need to differentiate with respect to x the RHS of relationship

(11) and we find

$$r^{*\prime\prime} = \frac{T_P' r^{*\prime}\left[T_A + T_P\right] - T_P\left[T_A'\left(1 - r^{*\prime}\right) + T_P' r^{*\prime}\right]}{\left(T_A + T_P\right)^2}$$
$$= \frac{T_P' r^{*\prime}\dfrac{T_A}{T_A + T_P} - \dfrac{T_P}{T_A + T_P}\left[T_A'\left(1 - r^{*\prime}\right)\right]}{T_A + T_P},$$

which can be rewritten

$$r^{*\prime\prime} = \frac{\left[T_P' - T_A'\right] r^{*\prime}\left(1 - r^{*\prime}\right)}{T_A + T_P}. \tag{12}$$

Relationship (12) shows that, as long as $T_P'(r^*(x)) < T_A'(s^*(x))$, then $r^{*\prime\prime}(x) < 0$. This means that, evaluated at the optimal allocations, the marginal absolute tolerance of the agent must be greater than the marginal absolute risk tolerance of the principal. The intuition behind this result is clear. We would expect that the more marginally risk tolerant the agent is, the better she can cope than the principal with the risk of not having the best distribution of results, especially if the agent has to assume all the effort's cost to obtain second-order stochastic dominant distributions. In other words, intuition tells us that agent's second-best optimal effort is less than what the principal would like her to exert when the agent's marginal absolute risk tolerance is higher than the principal's.

We note that the marginal tolerance is $T' = -\dfrac{A'}{A^2}$, where $A' = A[A - B]$, and $B = -\dfrac{u'''}{u''}$ is the absolute prudence (Kimball [1990]). Then, marginal tolerance can be written as $T' = \dfrac{B}{A} - 1$. Therefore, the condition $T_p' < T_A'$ is equivalent to $\dfrac{B_P}{A_P} < \dfrac{B_A}{A_A}$. This general formulation leads straight to the results of Hughes (1982) for the particular case in which preferences exhibit linear absolute risk tolerance. ■

3.3 TOSD and Moral Hazard

There are some other cases in which the principal's main goal may be to avoid bad states of nature at all cost. If this were the case, the principal can be called as downside risk averse (Menezes et al., 1980). An example of downside risk aversion in the applied economics literature is the case where preferences are loss averse. Loss aversion implies that losses have a greater impact on utility levels than gains of the same size (Prospect Theory of Kahneman and Tversky, 1979).

Downside risk aversion also implies that the principal is **prudent** ($G''' > 0$) and therefore, third-order stochastic dominant shifts provides the right ordering of the distribution of results. This leads us to the following proposition.

If an increase in agent's effort causes TOSD shifts on $F(\cdot \mid a)$, i.e. a shift that satisfies the property of $\int^x \int^y F_a(z \mid a)\, dydz \leq 0$, the principal will always want the agent to exert a greater effort than what the latter is willing to provide under second-best sharing rule, i.e. $\mu > 0$.

Proof: Everything remains the same as in Propositions 2 and 3, except that this time we have to incorporate TOSD shifts into the analysis. Consequently, we have to integrate by parts a third time the right hand side of relationship (7) and show that the resulting expression is **positive**. Then, integrating by parts the expression $\int_X \left[G_{xx}(r^*(x)) \int^x F_a(y \mid a)\, dy\right] dx$, we get

$$\begin{aligned}\int_X \left[G_{xx}(r^*(x)) \int^x F_a(y \mid a)\, dy\right] dx = G_{xx}(r^*(x)) \int^x \int^y F_a(z \mid a)\, dzdy \,\Big|_{\underline{x}}^{\overline{x}} \\ - \int_X \left[G_{xxx}(r^*(x)) \int^x \int^y F_a(z \mid a)\, dzdy\right] dx.\end{aligned} \tag{13}$$

Replacing (13) in (9) we obtain

$$\begin{aligned}\int_X G(x - s^*(x))\, f_a(x \mid a)\, dx = -G_x(r^*(\overline{x}))\, F_a^2(\overline{x} \mid a) + G_{xx}(r^*(\overline{x}))\, F_a^3(\overline{x} \mid a) \\ - \int_X \left[G_{xxx}(r^*(x)) \int^x \int^y F_a(z \mid a)\, dzdy\right] dx\end{aligned}$$

The definition of TOSD guarantees that $F_a^3(\overline{x} \mid a) \leq 0$ and $\int^x \int^y F_a(z \mid a)\, dzdy \leq 0$. Since $G_x(r^*(\overline{x}))$ is positive and we already know that $G_{xx}(r^*(\overline{x}))$ is negative, our focus is reduced to study the sign of $G_{xxx}(r^*(x))$. In order to prove Proposition 4, we only need that $G_{xxx}(r^*(x))$ to be positive. Note that

$$\begin{aligned}G_{xxx}(r^*(x)) = G'''(r^*(x)) \cdot (r^{*\prime}(x))^3 + \\ G''(r^*(x))\, r^{*\prime}(x) 2 r^{*\prime\prime}(x) + G''(r^*(x))\, r^{*\prime}(x)\, r^{*\prime\prime}(x) + G'(r^*(x))\, r^{*\prime\prime\prime}(x).\end{aligned}$$

From the previous equations and considering the sign of $r^{*\prime}(x)$, $r^{*\prime\prime}(x)$ obtained from Propositions 2 and 3 respectively, and also considering the properties of $G(r^*(x))$, that is $G' > 0$, $G'' < 0$ and $G''' > 0$, we observe that $G'''(r^*(x))\,(r^{*\prime}(x))^3$ and $G''(r^*(x))\, r^{*\prime}(x) 2 r^{*\prime\prime}(x)$ are positive, $G''(r^*(x))\, r^{*\prime}(x)\, r^{*\prime\prime}(x)$ is also positive, and therefore the sign of $G_{xxx}(r^*(x))$ is simply reduced to the sign of $r^{*\prime\prime\prime}(x)$.

To determine the sign of $r^{*\prime\prime\prime}$ we need to differentiate with respect to x relationship (12) in order to get

$$r^{*\prime\prime\prime} = \frac{\left(\left[T_P'' r^{*\prime} - T_A''(1-r^{*\prime})\right] r^{*\prime}(1-r^{*\prime}) + \left(T_P' - T_A'\right)\left[r^{*\prime\prime}(1-r^{*\prime}) - r^{*\prime} r^{*\prime\prime}\right]\right)(T_A + T_P)}{(T_A+T_P)^2} + \frac{-\left(T_P' - T_A'\right) r^{*\prime}(1-r^{*\prime})\left[T_P' r^{*\prime} - T_A'(1-r^{*\prime})\right]}{(T_A+T_P)^2}.$$

Factorization and using the definition of $r^{*\prime\prime}$ leads to

$$r^{*\prime\prime\prime}=\frac{\left[T_P^{\prime\prime}r^{*\prime}-T_A^{\prime\prime}\left(1-r^{*\prime}\right)\right]r^{*\prime}\left(1-r^{*\prime}\right)+\left(T_P^{\prime}-T_A^{\prime}\right)\left[r^{*\prime\prime}\left(1-r^{*\prime}\right)-r^{*\prime}r^{*\prime\prime}\right]}{T_A+T_P}+\frac{-r^{*\prime\prime}\left[T_P^{\prime}r^{*\prime}-T_A^{\prime}\left(1-r^{*\prime}\right)\right]}{T_A+T_P},$$

which is equal to

$$r^{*\prime\prime\prime}=\frac{\left[T_P^{\prime\prime}r^{*\prime}-T_A^{\prime\prime}\left(1-r^{*\prime}\right)\right]r^{*\prime}\left(1-r^{*\prime}\right)+\left(T_P^{\prime}-T_A^{\prime}\right)\left(1-2r^{*\prime}\right)+-r^{*\prime\prime}\left[T_P^{\prime}r^{*\prime}-T_A^{\prime}\left(1-r^{*\prime}\right)\right]}{T_A+T_P},$$

which in turn becomes

$$\begin{aligned}r^{*\prime\prime\prime} &= \frac{\left[T_P^{\prime\prime}r^{*\prime}-T_A^{\prime\prime}\left(1-r^{*\prime}\right)\right]r^{*\prime}\left(1-r^{*\prime}\right)+r^{*\prime\prime}\left[\left(T_P^{\prime}-T_A^{\prime}\right)\left(1-2r^{*\prime}\right)-\left[T_P^{\prime}r^{*\prime}-T_A^{\prime}\left(1-r^{*\prime}\right)\right]\right]}{T_A+T_P}\\ &= \frac{\left[T_P^{\prime\prime}r^{*\prime}-T_A^{\prime\prime}\left(1-r^{*\prime}\right)\right]r^{*\prime}\left(1-r^{*\prime}\right)+r^{*\prime\prime}\left[T_P^{\prime}-2T_A^{\prime}-3r^{*\prime}\left(T_P^{\prime}-T_A^{\prime}\right)\right]}{T_A+T_P},\end{aligned}$$

to finally result in

$$r^{*\prime\prime\prime}=\frac{\left[T_P^{\prime\prime}r^{*\prime}-T_A^{\prime\prime}\left(1-r^{*\prime}\right)\right]r^{*\prime}\left(1-r^{*\prime}\right)+r^{*\prime\prime}\left[T_P^{\prime}\left(1-3r^{*\prime}\right)-T_A^{\prime}\left(2-3r^{*\prime}\right)\right]}{T_A+T_P}. \tag{14}$$

Let us now analyze when relationship (14) is positive. Obviously, $r^{*\prime\prime\prime}$ is positive when the numerator of the RHS relationship (14) is positive, but that does not give us much intuition about the result. Let us first take a look at $r^{*\prime\prime}\left[T_P^{\prime}\left(1-3r^{*\prime}\right)-T_A^{\prime}\left(2-3r^{*\prime}\right)\right]$. As $r^{*\prime\prime}<0$ is negative, we may expect $\left[T_P^{\prime}\left(1-3r^{*\prime}\right)-T_A^{\prime}\left(2-3r^{*\prime}\right)\right]$ to be negative as well. From proposition 3 we know that $T_P^{\prime}<T_A^{\prime}$ so, as long as $\left(1-3r^{*\prime}\right)$ is positive, it is guaranteed that $\left[T_P^{\prime}\left(1-3r^{*\prime}\right)-T_A^{\prime}\left(2-3r^{*\prime}\right)\right]$ will be negative, and then $r^{*\prime\prime}\left[T_P^{\prime}\left(1-3r^{*\prime}\right)-T_A^{\prime}\left(2-3r^{*\prime}\right)\right]$ will be positive. Note that, as long as $0<r^{*\prime}<\frac{2}{3}$, we guarantee that $r^{*\prime\prime}\left[T_P^{\prime}\left(1-3r^{*\prime}\right)-T_A^{\prime}\left(2-3r^{*\prime}\right)\right]$ will be positive.

Let us now examine $\left[T_P^{\prime\prime}r^{*\prime}-T_A^{\prime\prime}\left(1-r^{*\prime}\right)\right]r^{*\prime}\left(1-r^{*\prime}\right)$ and observe that this expression is positive when $T_P^{\prime\prime}r^{*\prime}-T_A^{\prime\prime}\left(1-r^{*\prime}\right)$ is positive. Let us rewrite the expression $T_P^{\prime\prime}r^{*\prime}-T_A^{\prime\prime}\left(1-r^{*\prime}\right)$ as $\frac{\partial\left(T_P^{\prime}-T_A^{\prime}\right)}{\partial x}$ and, from relationship (14) observe that, as long as $\frac{\partial\left(T_P^{\prime}-T_A^{\prime}\right)}{\partial x}>0$, we obtain $r^{*\prime\prime\prime}(x)>0$. This means that as the payoff grows the marginal risk tolerance of the principal increases at a higher rate than the marginal risk tolerance of the agent. This is intuitive since we know from Proposition 3 that $T_p^{\prime}<T_A^{\prime}$ and, as the output grows (and both receive more resources to share), the speed of change is likely to be higher for the principal because he starts from a lower initial level. ■

The analysis presented here provides some sufficient conditions for $r^{*\prime\prime\prime}$ to be positive. However, we know that it is enough for the numerator of relationship (14) to be positive in order to guarantee the result, which is a less restrictive condition.

3.4 N^{th}OSD and Moral Hazard

In the previous sections we have discussed specific conditions under which different types of stochastic dominant shifts in the distribution of results occur when the agent exerts more effort. We have provided some economic intuition for the cases of FOSD, SOSD and TOSD. Analyzing higher order stochastic dominant shifts is a much more challenging task. For example, economic intuition for FOOSD is likely to be connected with aversion to outer risk (Menezes and Wang, 2005), but the algebraic tractability of the proof becomes cumbersome.

In the above sections we explicitly studied the relationship between FOSD, SOSD, TOSD and the function $\int_X G(r^*(x)) f_a(x \mid a)\, dx$. Define $V(x) \equiv G(r^*(x))$ and observe that since when the principal is N^{th} order risk averse, we have $(-1)^k G^{(k)} < 0$ for $k = 1, ..., N$. Aforementioned, an increase in agent's effort a that represents a NOSD shift on $F(\cdot \mid a)$ is equivalent to $\int_{\underline{x}}^{\overline{x}} V(x) dF_a(x \mid a) > 0$, if and only if $(-1)^k V^{(k)} < 0$ for $k = 1, ..., N$. It remains to examine the sign of $(-1)^n r^{*(n)}(x)$, which is the only other element that appears in the iterative integrals. We now ready to state the main result of the paper.

Proposition 4

Assume that function G and r^* are C^N differentiable functions that satisfy

$$\begin{aligned} (-1)^n G^{(n)}(x) &< 0 \\ (-1)^n (r^*)^{(n)}(x) &< 0, \end{aligned}$$

for all $x \in X$ and all integer $n \leq N$. Then for all $x \in X$ and all integer $n \leq N$, we have

$$(-1)^n V^{(n)}(x) < 0,$$

so that an increase in agent's effort causes N^{th}OSD shifts on $F(\cdot \mid a)$, the principal will always want the agent to exert a greater effort than what the latter is willing to provide under second-best sharing rule, i.e. $\mu > 0$.

Proof: The proof relies on the Faà di Bruno formula[4] (1855 and 1857) that states that for

[4] The Faà di Bruno formula (1855 and 1857) provides an explicit expression the derivative of order n of a composite function and generalizes the chain rule.

function $V = G(r^*)$, for all $n \in \mathbb{N}$,

$$V^{(n)}(x) = \sum_{j=0}^{n} \frac{n!}{b_1!...b_n!} G^{(j)}(r^*(x)) \prod_{i=1}^{n} \left(\frac{(r^*)^{(i)}(x)}{i!} \right)^{b_i},$$

with $b_i \in \mathbb{N}$ and

$$\begin{aligned} b_1 + ... + b_n &= j \\ b_1 + 2b_2 ... + nb_n &= n. \end{aligned}$$

Let assume that for all $x \in X$ and all $i \in \mathbb{N}^*$, $(-1)^i (r^*)^{(i)}(x) < 0$. It follows that

$$\begin{aligned} V^{(n)}(x) &= \sum_{j=0}^{n} \frac{n!}{b_1!...b_n!} G^{(j)}(r^*(x)) \prod_{i=1}^{n} \left(\frac{(-1)^{i+1}(r^*)^{(i)}(x)}{i!} (-1)^{i+1} \right)^{b_i} \\ &= \sum_{j=0}^{n} \frac{n!}{b_1!...b_n!} G^{(j)}(r^*(x)) \prod_{i=1}^{n} \left(\frac{(-1)^{i+1}(r^*)^{(i)}(x)}{i!} \right)^{b_i} \prod_{i=1}^{n} [(-1)]^{(i+1)b_i} \\ &= \sum_{j=0}^{n} \frac{n!}{b_1!...b_n!} G^{(j)}(r^*(x)) \prod_{i=1}^{n} \left(\frac{(-1)^{i+1}(r^*)^{(i)}(x)}{i!} \right)^{b_i} [(-1)]^{j+n}, \end{aligned}$$

using the conditions on integers b_i. Thus

$$V^{(n)}(x) = (-1)^n \sum_{j=0}^{n} \frac{n!}{b_1!...b_n!} (-1)^j G^{(j)}(r^*(x)) \prod_{i=1}^{n} \left(\frac{(-1)^{i+1}(r^*)^{(i)}(x)}{i!} \right)^{b_i}.$$

Since by assumption $(-1)^{i+1}(r^*)^{(i)}(x) > 0$ and $(-1)^j G^{(j)}(r^*(x)) < 0$ for all $j \in \mathbb{N}^*$, we deduce $(-1)^n V^{(n)}(x) < 0$ for all $x \in X$. ■

Of course, Proposition 3 does not provide a necessary/sufficient condition on the primitives of the model to ensure we are not assure that . In fact, as developed below looking at the some classic classes of utility functions, it is not always the case.

4 Examples

4.1 CARA Utility Functions

Let $G(x) = \frac{1-e^{-bx}}{b}$ and $u(x) = \frac{1-e^{-ax}}{a}$, with a and b positive constants. For this class of utility functions, for the principal (resp. agent), we have $A_n(x) = b$ (resp. a); in particular, the tolerance is constant, so the marginal tolerance is equal to zero. It is easy to verify that Furthermore, it is easy to verify that G' and u' are the Laplace transforms of the distributions $F_G(s) = H(s-b)$ and $F_u(s) = H(s-a)$ respectively, where H denotes the Heaviside function. Since we focus on interior solutions, we take $X = \mathbb{R}$. For this class of preferences, the coefficient of risk aversion of order n is constant, for all $n \in \mathbb{N}$.

The first order condition for the Pareto allocation is

$$\frac{G'(s - s^*(x))}{u'(s^*(x))} = e^{-bx+(a+b)s^*(x)} = \lambda > 0,$$

so that the sharing rule s^* is given by

$$s^*(x) = \frac{b}{a+b}x + \frac{\ln\lambda}{a+b}$$

and therefore is of the affine type in x.

4.2 CRRA Utility Functions

Let $G(x) = \frac{x^{1-b}-1}{1-b}$ (resp. $\ln x$) if $b \neq 1$ (resp. $b = 1$) and $u(x) = \frac{x^{1-a}-1}{1-a}$ (resp. $\ln x$) if $a \neq 1$ (resp. $a = 1$) , with a and b positive constants. In this case, we have $X = \mathbb{R}_+$. For this class of utility functions, for the principal (resp. agent), we have $A_n(x) = \frac{b+n-1}{x}$ (resp. $\frac{a+n-1}{x}$); in particular, the tolerance is linear, so the marginal tolerance is constant. Furthermore, it is easy to verify that G' and u' are the Laplace transforms of the distributions $F_G(s) = \frac{s^b}{\Gamma(b+1)}$ and $F_u(s) = \frac{s^a}{\Gamma(a+1)}$ respectively, where Γ denotes Euler's Gamma function. The first order condition for the Pareto allocation is

$$\lambda(x - s(x))^b = (s(x))^a, \tag{15}$$

for all $x \in \mathbb{R}_+$ and $s(x) \in [0, x]$, where we have used s instead of s^* the to lighten the notation.

4.2.1 Case $a = 1$ and $b = 2$

A closed form solution is available and it is easy to verify that

$$\begin{aligned} s(x) &= \frac{1 + 2\lambda x - \sqrt{1 + 4\lambda x}}{2\lambda} \\ r(x) &= \frac{\sqrt{1 + 4\lambda x} - 1}{2\lambda}. \end{aligned}$$

It follows that

$$\begin{aligned} s'(x) &= 1 - \frac{1}{1+4\lambda x} > 0 \\ s''(x) &= \frac{2\lambda}{(1+4\lambda x)^{\frac{3}{2}}} > 0, \end{aligned}$$

and more generally

$$s^{(n)}(x) = (-1)^n \times 1 \times 3 \times ... \times (2n-3)(2\lambda)^{n-1}(1+4\lambda x)^{-\frac{2n-1}{2}}.$$

Hence we have

$$\begin{aligned} (-1)^{n+1}s^{(n)}(x) &< 0 \\ (-1)^{n+1}r^{(n)}(x) &> 0. \end{aligned}$$

4.2.2 Case $a = 2$ and $b = 1$

A closed form solution is available and it is easy to verify that

$$\begin{aligned} s(x) &= \frac{-\lambda + \sqrt{\lambda^2 + 4\lambda x}}{2} \\ r(x) &= \frac{2x + \lambda - \sqrt{\lambda^2 + 4\lambda x}}{2}. \end{aligned}$$

It follows that

$$\begin{aligned} s'(x) &= \frac{\lambda}{\sqrt{\lambda^2 + 4\lambda x}} > 0 \\ s''(x) &= -\frac{2\lambda^2}{(\lambda^2 + 4\lambda x)^{\frac{3}{2}}} < 0, \end{aligned}$$

and more generally

$$s^{(n)}(x) = (-1)^{n+1} \times 1 \times 3 \times ... \times (2n-3)2^{n-1}\lambda^n(\lambda^2 + 4\lambda x)^{-\frac{2n-1}{2}}.$$

Hence we have

$$\begin{aligned} (-1)^{n+1}s^{(n)}(x) &> 0 \\ (-1)^{n+1}r^{(n)}(x) &< 0. \end{aligned}$$

4.2.3 General case

Totally differentiating relationship (15) with respect to x leads to

$$a\frac{s'(x)}{s(x)} = b\frac{1-s'(x)}{x-s(x)},$$

i.e.,

$$s'(x) = \frac{bs(x)}{a(x-s(x))+bs(x)} \in (0,1)\,.$$

Again, differentiating with respect to x and rearranging terms yields

$$s''(x) = (b-a)ab\frac{s(x)(x-s(x))}{\left[a(x-s(x))+bs(x)\right]^3}.$$

It follows that s'' has a constant sign with $s''>0$ (resp. $s''<0$) iff $b>a$ (resp. $a<b$), i.e. if the principal is more (resp. less) risk averse than the agent. Observe

$$\begin{aligned} s''(x) &= (b-a)\frac{(s'(x))^2(1-s'(x))}{bs(x)} \\ &= \frac{b-a}{abx}s'(x)(1-s'(x))\left(b(1-s'(x))+as'(x)\right), \end{aligned}$$

and differentiating one more time with respect to x after rearranging terms we find that

$$s'''(x) = \frac{(s''(x))^2}{(b-a)s'(x)(1-s'(x))}(-2a^2(x-s(x))-2b^2s(x)+abx).$$

Set $\varpi(x) = \frac{s(x)}{x} \in (0,1)$. We want to study the sign of

$$\varphi(x) = -2a^2(1-\varpi(x)) - 2b^2\varpi(x) + ab.$$

Next, note that for all $x>0$, $\varpi(x)$ is implicitly defined by

$$\lambda x^{b-a}(1-\varpi(x))^b = (\varpi(x))^a.$$

It is easy to verify that when $b>a$, ϖ is a strictly increasing function with $\lim_{0^+}\varpi = 0$ and $\lim_{\infty}\varpi = 1$. Conversely, when $b<a$, ϖ is a strictly decreasing function with $\lim_{0^+}\varpi = 1$ and $\lim_{\infty}\varpi = 0$.

Case $b>a$: Clearly, function φ is decreasing in x with $\varphi(0) = a(b-2a)$ and $\lim_{\infty}\varphi =$

$b(a-2b)<0$. We conclude that φ has a constant sign, namely negative, if and only if $a<b\leq 2a$. In this latter case, we have $s''>0$ and $s'''<0$.

Case $b<a$: Function φ is again decreasing in x with $\varphi(0)=b(a-2b)$ and $\lim_{\infty}\varphi = a(b-2a)<0$. We conclude that φ has a constant sign, namely negative, if and only if $b<a\leq 2b$. In this latter case, we have $s''<0$ and $s'''>0$. To sum up,

$$(-1)^3 r'''(x)<0 \text{ for all } x\geq 0 \text{ iff } a<b\leq 2a.$$

We conclude that, in general, s''' does not have a constant sign. We now provide a full characterization for the general case.

Proposition 5 *For all $x\in\mathbb{R}_{++}$, we have*

$$(-1)^{n+1} r^{(n)}(x)\geq 0 \text{ iff } a\leq b\leq 2a.$$

We have already seen that this is a necessary condition. Then, note that

$$\lambda^{\frac{1}{b-a}} x = (\varpi(x))^{\frac{a}{b-a}}(1-\varpi(x))^{-\frac{b}{b-a}}, \tag{16}$$

so that

$$\begin{aligned} r(x) &= x(1-\varpi(x)) \\ &= \lambda^{-\frac{1}{b-a}}\left(\frac{\varpi(x)}{1-\varpi(x)}\right)^{\frac{a}{b-a}}. \end{aligned}$$

Set $\alpha=\frac{b-a}{a}$ and $Y(x)=\left(\frac{\varpi(x)}{1-\varpi(x)}\right)^{\frac{a}{b-a}}$ and so that

$$r(x)=\lambda^{-\frac{1}{b-a}} Y(x).$$

Observe that by assumption $\alpha\in[0,1]$. Solving for $\varpi(x)$ leads to

$$\varpi(x)=\frac{Y(x)}{1+Y(x)},$$

and plugging back into relationship (16), we find that for all $x\in\mathbb{R}_+$

$$Y(x)(1+Y^{\alpha}(x))=\lambda^{\frac{1}{b-a}} x.$$

Next, we want that show that when $\alpha \in [0,1]$, $(-1)^{n+1}Y^{(n)}(x) \geq 0$. Define auxiliary f with

$$f(x) = x(1+x^{\alpha}).$$

Note that

$$\begin{aligned} f'(x) &= 1+(1+\alpha)x^{\alpha} > 0 \\ f''(x) &= (1+\alpha)\alpha x^{\alpha-1} > 0, \end{aligned}$$

and more generally for $n \in \mathbb{N}^*$

$$(-1)^n f^{(n)}(x) > 0.$$

Then, define function g as the inverse of f, i.e. $g = f^{-1}$. By Ostrowski's formula, we have

$$g^{(n)}(x) = \sum_{j=0}^{n-1} \frac{(-1)^j}{(f'(x))^{n+j}}(n+j-1)! \prod_{i=2}^{n} \frac{1}{b_i!}\left(\frac{f^{(i)}(x)}{i!}\right)^{b_i}, \tag{17}$$

with $b_i \in \mathbb{N}$ and

$$\begin{aligned} b_2 + ... + b_n &= j \\ b_1 + 2b_2 ... + nb_n &= n+j-1. \end{aligned}$$

Manipulating relationship (17) leads to

$$\begin{aligned} g^{(n)}(x) &= \sum_{j=0}^{n-1} \frac{(-1)^j}{(f'(x))^{n+j}}(n+j-1)! \prod_{i=2}^{n} \frac{1}{b_i!}\left(\frac{(-1)^i f^{(i)}(x)}{i!}\right)^{b_i} \times (-1)^{ib_i} \\ &= \sum_{j=0}^{n-1} \frac{(-1)^{n-1}}{(f'(x))^{n+j}}(n+j-1)! \prod_{i=2}^{n} \frac{1}{b_i!}\left(\frac{(-1)^i f^{(i)}(x)}{i!}\right)^{b_i}, \end{aligned}$$

as $\sum_{i=2}^{n} ib_i = n+j-1$. Since for all $i \in \mathbb{N}^*$, we have $(-1)^i f^{(i)}(x) > 0$, we easily conclude that for all $x \in \mathbb{R}_+$ and all $n \in \mathbb{N}^*$, $(-1)^{n+1} g^{(n)}(x) > 0$. This concludes the proof. ■

In the case of interior solutions, Proposition 5 extends to the class of HARA utility functions. To see this, when $G(x) = \frac{(x+B)^{1-b}-1}{1-b}$ and $u(x) = \frac{(x+A)^{1-a}-1}{1-a}$, the FOC is

$$\lambda(x+B-s(x))^b = (A+s(x))^a.$$

Set $S(y) = A + s(y - (A + B))$, with $y = x + A + B$, so that

$$\lambda(y - S(y))^b = (S(y))^a.$$

Since only affine transformations are involved, the analysis conducted and the results obtained for the CRRA case still apply. **This case is a generalization of the result by Hughes (1982) for the case $n = 2$.**

IS THIS TRUE?

5 Conclusion

In this paper we build a bridge between some developments in the economic theory of risk and uncertainty with the classic moral hazard problem. We study the conditions under which an increase in the level of effort exerted by the agent induces higher order stochastic dominant shifts in the distribution of results. We show that in the second-best equilibrium higher order derivatives of the principal's share r^* are key conditions for our results for N^{th} order stochastic dominance.

We argue that there are some contexts in which second-order or third-order stochastic dominance may provide the right ordering of distributions. We provide the economic intuition and also the mathematical conditions under which second and third-order stochastic dominance makes sense in moral hazard problems.

We believe that, for instance in the case of delegated portfolio management, one can find situations where a principal may want the agent to minimize risk (second-order risk decision) or alternatively, the principal may want the agent to avoid losses of any kind (loss aversion, prudent behavior or third-order risk decision). In all these cases, first-order stochastic dominance is not enough to rank the distributions of results and consequently, it is important to consider higher-order stochastic dominance and higher-order risk attitudes in some specific moral hazard contexts.

6 References

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